NASA Technical Memorandum 89851

A Finite Difference Scheme for Three-Dimensional Steady Laminar Incompressible Flow

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Prepared for the
5th International Conference on Numerical Methods in Laminar and Turbulent Flow
cosponsored by Concordia University, P&W Canada, the International Journal for Numerical Methods in Fluids, and the Natural Sciences and Research Council of Canada
Montreal, Canada, July 6–10, 1987

NASA

E-3513

A FINITE DIFFERENCE SCHEME FOR THREE-DIMENSIONAL STEADY LAMINAR INCOMPRESSIBLE FLOW

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ABSTRACT

A finite difference scheme for three-dimensional steady laminar incompressible flows is presented. The Navier-Stokes equations are expressed conservatively in terms of velocity and pressure increments (delta form). First order upwind differences are used for first order partial derivatives of velocity increments resulting in a diagonally dominant matrix system. Central differences are applied to all other terms for second order accuracy. The SIMPLE pressure correction algorithm is used to satisfy the continuity equation. Numerical results are presented for cubic cavity flow problems for Reynolds numbers up to 2000 and are in good agreement with other numerical results.

1. INTRODUCTION

Two basic problems for a finite difference scheme for incompressible Navier-Stokes equations are stability and accuracy. Second order accuracy can be achieved by using a central difference method, but it is unstable at high Reynolds number. Stability at very high Reynolds numbers can be achieved using first order upwind differencing, but it is only first order accurate. Several schemes which combine upwind and central differences have been developed [1], but these methods are only first order accurate. The second order upwind method encounters difficulties with convergence and accuracy [2].

The present approach retains the second order accuracy of central differencing while utilizing the stability of first order upwind differencing. The incompressible Navier-Stokes equations in conservative delta form [3] are formulated, i.e., momentum equations are written in terms of the increments of primitive variables rather than primitive variables themselves. The equations are then decoupled and solved semi-implicitly. By using central differencing for the primitive variable terms on the right hand side of the equations, the scheme achieves second order accuracy. By applying first order upwind differencing for the first order partial derivative velocity increment terms on the left hand side of the equations, the method yields a diagonally dominant matrix system which is then solved by a line by line tridiagonal-matrix algorithm. To satisfy the continuity equation, the SIMPLE algorithm [1,4,5] for pressure correction is used.

The present finite difference approach is used to solve the cubic cavity flow problem which was studied by numerous authors [6-10]. This problem is important because it is a three-dimensional steady separated flow with very simple geometry and boundary conditions. In this paper, the results are compared with those of other numerical methods for Reynolds numbers up to 2000.

2. GOVERNING EQUATIONS

The Navier-Stokes equations for incompressible viscous flows are:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \tag{1}$$

$$\frac{\partial u}{\partial t} + \frac{\partial}{\partial x}(uu) + \frac{\partial}{\partial y}(vu) + \frac{\partial}{\partial z}(wu) = -\frac{\partial p}{\partial x} + \frac{1}{Re}\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right) \tag{2}$$

$$\frac{\partial v}{\partial t} + \frac{\partial}{\partial x}(uv) + \frac{\partial}{\partial y}(vv) + \frac{\partial}{\partial z}(wv) = -\frac{\partial p}{\partial y} + \frac{1}{Re}\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2}\right) \tag{3}$$

$$\frac{\partial w}{\partial t} + \frac{\partial}{\partial x}(uw) + \frac{\partial}{\partial y}(vw) + \frac{\partial}{\partial z}(ww) = -\frac{\partial p}{\partial z} + \frac{1}{Re}\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2}\right) \tag{4}$$

where u, v, and w are velocity components, p is pressure, and Re is Reynolds number.

All variables are nondimensionalized by the characteristic velocity of the upper surface and the characteristic length of the box side.

Let u^N, v^N, w^N, and p^N be the quantities corresponding to u, v, w, and p at time level N. Let

$$\delta u^N = u^{N+1} - u^N \tag{5}$$

and let δv^N, δw^N, and δp^N be defined similarly.

Applying the fully implicit method to Eq. (2) for u^{N+1}, one obtains:

$$\frac{\partial u^N}{\partial t} + \frac{\partial}{\partial x}(u^N + \delta u^N)(u^N + \delta u^N) + \frac{\partial}{\partial y}(v^N + \delta v^N)(u^N + \delta u^N)$$

$$+ \frac{\partial}{\partial z}(w^N + \delta w^N)(u^N + \delta u^N) = -\frac{\partial}{\partial x}(p^N + \delta p^N)$$

$$+ \frac{1}{Re}\left[\frac{\partial^2}{\partial x^2}(u^N + \delta u^N) + \frac{\partial^2}{\partial y^2}(u^N + \delta u^N) + \frac{\partial^2}{\partial z^2}(u^N + \delta u^N)\right] \tag{6}$$

After dropping all second order terms and rearranging, Eq. (6) becomes

$$\frac{\partial u^N}{\partial t} + \frac{\partial}{\partial x}(2u^N\delta u^N) + \frac{\partial}{\partial y}(v^N\delta u^N) + \frac{\partial}{\partial z}(w^N\delta u^N)$$

$$-\frac{1}{Re}\left(\frac{\partial^2\delta u^N}{\partial x^2} + \frac{\partial^2\delta u^N}{\partial y^2} + \frac{\partial^2\delta u^N}{\partial z^2}\right) = -\left[\frac{\partial}{\partial x}(u^N u^N) + \frac{\partial}{\partial y}(v^N u^N)\right.$$

$$\left. + \frac{\partial}{\partial z}(w^N u^N)\right] - \frac{\partial p^N}{\partial x} + \frac{1}{Re}\left(\frac{\partial^2 u^N}{\partial x^2} + \frac{\partial^2 u^N}{\partial y^2} + \frac{\partial^2 u^N}{\partial z^2}\right)$$

$$-\frac{\partial}{\partial x}\delta p^N - \frac{\partial}{\partial y}u^N\delta v^N - \frac{\partial}{\partial z}u^N\delta w^N \tag{7a}$$

The above equation is decoupled from the others by setting δv^N, δw^N, and δp^N equal to zero. With this assumption, the right hand side of the equation is exactly the steady state incompressible viscous equation. The accuracy of the finite difference method applied to this expression is the accuracy of the scheme. Notice that the usual linearization gives the term $\partial/\partial x\,(u^N\delta u^N)$ instead of $\partial/\partial x\,(2u^N\delta u^N)$ as above.

For completeness, the corresponding expressions for Eqs. (3) and (4) are

$$\frac{\partial v^N}{\partial t} + \frac{\partial}{\partial x}(u^N \delta v^N) + \frac{\partial}{\partial y}(2v^N \delta v^N) + \frac{\partial}{\partial z}(w^N \delta v^N) - \frac{1}{Re}\left(\frac{\partial^2 \delta v^N}{\partial x^2} + \frac{\partial^2 \delta v^N}{\partial y^2} + \frac{\partial^2 \delta v^N}{\partial z^2}\right) = -\left[\frac{\partial}{\partial x}(u^N v^N) + \frac{\partial}{\partial y}(v^N v^N) + \frac{\partial}{\partial z}(w^N v^N)\right] - \frac{\partial p^N}{\partial y} + \frac{1}{Re}\left(\frac{\partial^2 v^N}{\partial x^2} + \frac{\partial^2 v^N}{\partial y^2} + \frac{\partial^2 v^N}{\partial z^2}\right) \qquad (7b)$$

$$\frac{\partial w^N}{\partial t} + \frac{\partial}{\partial x}(u^N \delta w^N) + \frac{\partial}{\partial y}(v^N \delta w^N) + \frac{\partial}{\partial z}(2w^N \delta w^N) - \frac{1}{Re}\left(\frac{\partial^2 \delta w^N}{\partial x^2} + \frac{\partial^2 \delta w^N}{\partial y^2} + \frac{\partial^2 \delta w^N}{\partial z^2}\right) = -\left[\frac{\partial}{\partial x}(u^N w^N) + \frac{\partial}{\partial y}(v^N w^N) + \frac{\partial}{\partial z}(w^N w^N)\right] - \frac{\partial p^N}{\partial z} + \frac{1}{Re}\left(\frac{\partial^2 w^N}{\partial x^2} + \frac{\partial^2 w^N}{\partial y^2} + \frac{\partial^2 w^N}{\partial z^2}\right) \qquad (7c)$$

Let u^* be an estimate for u^{N+1} etc., then the velocity field (u^*, v^*, and w^*) does not satisfy continuity in general. To correct velocity and pressure, the following pressure equation, based on the SIMPLE pressure correction algorithm [12], is solved.

$$\frac{\partial^2 \delta p^N}{\partial x^2} + \frac{\partial^2 \delta p^N}{\partial y^2} + \frac{\partial^2 \delta p^N}{\partial z^2} = \frac{1}{\Delta t}\left(\frac{\partial u^*}{\partial x} + \frac{\partial v^*}{\partial y} + \frac{\partial w^*}{\partial z}\right) \qquad (8)$$

where δp^N is the pressure correction. Velocity and pressure are then corrected as follows:

$$p^{N+1} = p^N + \alpha_p \, \delta p^N \qquad (9)$$

$$u^{N+1} = u^* - \Delta t \frac{\partial \delta p^N}{\partial x} \qquad (10)$$

$$v^{N+1} = v^* - \Delta t \frac{\partial \delta p^N}{\partial y} \qquad (11)$$

$$w^{N+1} = w^* - \Delta t \frac{\partial \delta p^N}{\partial z} \qquad (12)$$

where α_p is an under-relaxation factor [12].

3. NUMERICAL METHOD

An equally spaced rectangular coordinate system is used throughout this study. For the first order space derivatives of the left hand side of Eq. (7), first order upwind differencing is used, e.g.,

$$\frac{\partial}{\partial x}\left(2u^N \delta u^N\right) = \begin{cases} \dfrac{2\left(u^N_{i,j,k}\delta u^N_{i,j,k} - u^N_{i-1,j,k}\delta u^N_{i-1,j,k}\right)}{\Delta x} & \text{when } u^N_{i,j,k} \geq 0 \\[2ex] \dfrac{2\left(u^N_{i+1,j,k}\delta u^N_{i+1,j,k} - u^N_{i,j,k}\delta u^N_{i,j,k}\right)}{\Delta x} & \text{when } u^N_{i,j,k} < 0 \end{cases} \tag{13}$$

Similar expressions apply to

$$\frac{\partial}{\partial y}\left(v^N \delta u^N\right) \quad \text{and} \quad \frac{\partial}{\partial z}\left(w^N \delta u^N\right)$$

Central differencing is used for all diffusion terms, e.g.,

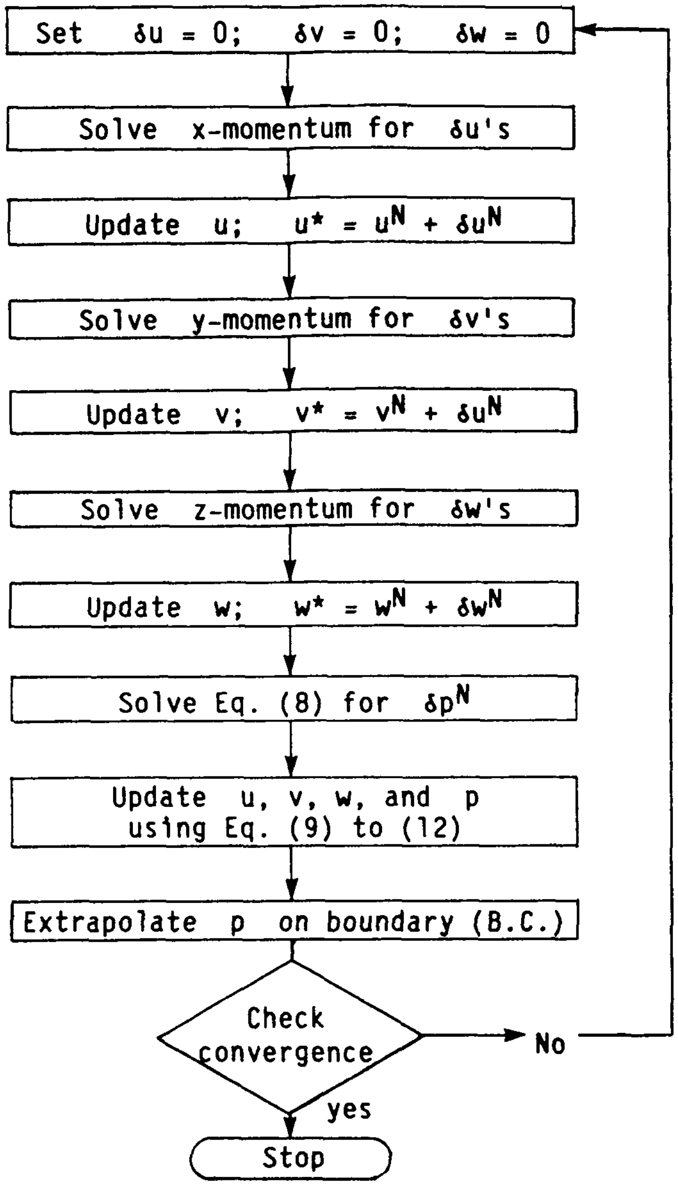
$$\frac{\partial^2 \varphi}{\partial x^2} = \frac{\varphi_{i+1,j,k} - 2\varphi_{i,j,k} + \varphi_{i-1,j,k}}{\Delta x^2}$$

where φ can be any variable.

All terms on the right hand side of Eq. (7) are discretized by using central differencing, e.g.,

$$\frac{\partial}{\partial x}\left(u^N u^N\right) = \frac{u^N_{i+1,j,k}u^N_{i+1,j,k} - u^N_{i-1,j,k}u^N_{i-1,j,k}}{2\Delta x}$$

A line by line tridiagonal-matrix algorithm is used. In the x-momentum equation, Eq. 7(a), δu^N is found by sweeping implicity in the z direction only once. Similarly, δv^N and δw^N are found by sweeping implicity in the x and y directions respectively. The Poisson equation for δp^N, Eq. (8), is swept in all three directions (ADI method). Central differences are used for Eqs. (8) to (12). The calculation procedure is sketched schematically as follows:



4. INITIAL AND BOUNDARY CONDITIONS

For the initial iteration, u, v, w, and p are set to zero at each grid point, except $u = 1$ on $z = 1$ (upper surface); all values on the boundary remain unchanged, except the pressure which is adjusted by second order extrapolation at each iteration.

5. RESULTS

All results presented here use the convergence criteria of 0.0001 for maximum δu with an underrelaxation factor for pressure $\alpha_p = 0.8$. All grid sizes are 30 by 30 by 30 except otherwise indicated.

The geometry and the boundary conditions for cubic cavity flow are shown in Fig. 1. A cubical box with its upper surface moving at constant speed $u = 1$ is shown in the figure.

The results for Reynolds number 100 are given in Figs. 2 to 5. Figure 2 shows the comparison of the velocity profile on the vertical center line with those of Cazalbou et al. [8] and Goda [6]. The agreement with both results is excellent.

In order to show the flow directions for very small velocity vectors, all velocity vectors are stretched to constant magnitude (called velocity field in Ref. [8]). The velocity field in the symmetry plane is illustrated for $Re = 100$ in Fig. 3. A primary vortex is clearly shown in the figure and a small corner eddy at the bottom corner A, is also visible. Figure 4 shows a larger corner eddy at corner A, on the cross section just next to the side wall.

Figures 5(a) to (d) show secondary flows on cross sections from $x = 0.63$ back to $x = 0.47$. A pair of large vortices are observed in reverse flow under the primary vortex. The sequence of motion for these vortices are shown from Figs. 5(a) to (d). This pair of vortices is formed from the side wall and moves toward the symmetry plane over a very short distance. They are mixed and dispersed near the symmetry plane. Another pair of secondary vortices is formed from the side walls close to the moving top surface (Fig. 5(c)) and disperses in a very short distance.

The results for $Re = 400$, 1000, and 2000 are shown in Figs. 6 to 9. The results are compared with other methods in Figs. 6(a) to (c). Figure 6(a) shows the excellent agreement with the results given by Cazalbou et al. [8] for $Re = 400$. The results given by Goda [6] in the same figure, however, are slightly different. For the case of $Re = 1000$, agreement with the results of Cazalbou et al. are very good except for only one point near the bottom surface. For $Re = 2000$, excellent agreement with the results given by Cazalbou et al. [8] is shown in Fig. 6(c).

Velocity vectors in the symmetry plane for $Re = 400$ are shown in Fig. 7(a). The variation of magnitude of the velocity vectors is shown in this figure. Some of the magnitudes of velocity vectors are as small as 0.0002. Figure 7(b) is identical to Fig. 7(a), except it shows only the direction of the velocity vectors. Notice that the eddy at corner A, for $Re = 400$ is larger than that for $Re = 100$. In Figs. 7(c) and (d), velocity fields in the symmetry plane for $Re = 1000$ and 2000 are shown. In these figures, the location of the primary vortex center is closer to the center of the symmetry plane. A larger eddy is clearly seen at the corner B. Velocity fields in the vertical plane next to the side wall for Re = 400, 1000, and 2000 are given in Figs. 8(a) to (c). They are very different from the case for Re = 100 in Fig. 4. A larger pair of upper secondary vortices for Re = 400, 1000, and 2000 are observed in Figs. 9(a) to (c). In Fig. 9(c), there are three pairs of vortices close to the bottom surface instead of one pair seen for the cases of Re = 400 and 1000. It is similar to the phenomena reported experimentally by Koseff et al. [11] for Re = 2000. These vortices only appear in a very short distance around x = 0.54.

The convergence history for Re = 100 is illustrated in Fig. 10. CPU-time is only 19.8 sec on a CRAY X/MP computer for the 20 by 20 by 20 grid size. CPU-time on a CRAY X/MP for different cases are given in Table 1. It shows that the present scheme is fairly fast and efficient for low Reynolds numbers. However, when Reynolds numbers are high (greater than 1000), the scheme needs some improvement as shown by cpu-time in Table 1.

6. CONCLUSION

A diagonally dominant finite difference scheme with second order accuracy is presented. Comparisons of present results with other numerical methods are in very good agreement. CPU-time for the case of Re = 100 with 20 by 20 by 20 grid points is only 19.8 sec on a CRAY X/MP computer.

ACKNOWLEDGMENTS

The authors wish to thank Drs. John Goodrich, H.C. Kao, and M.S. Liou for many useful discussions.

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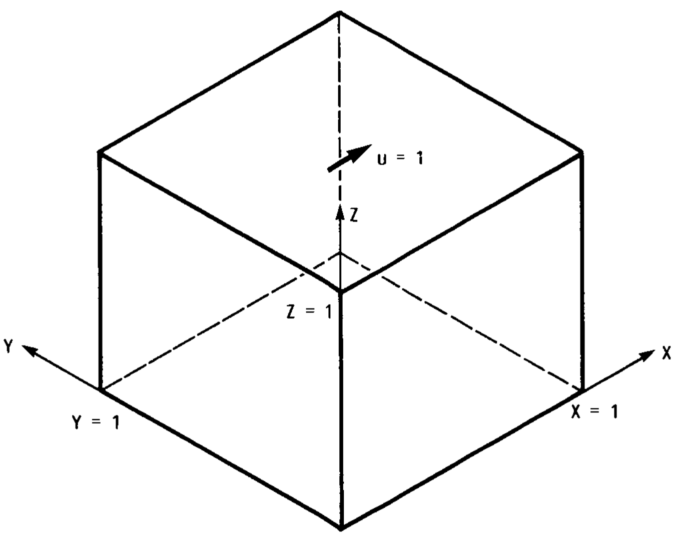
TABLE 1. - COMPUTING CPU-TIME

[CRAY X/MP, error <0.0001.]

Reynolds number	Grid size	ΔT	CPU-time, sec	Number of iterations
100	20 by 20 by 20	0.1	19.80	162
100	30 by 30 by 30	.1	88.39	230
400	30 by 30 by 30	.1	108.71	286
1000	30 by 30 by 30	.1	147.97	393
2000	50 by 50 by 50	.01	4712.76	2815



B.C. { u = 1 UPPER SURFACE; u = 0 OTHER SURFACES
v = 0 ALL SURFACES
w = 0 ALL SURFACES
P SECOND-ORDER EXTRAPOLATION TO ALL SURFACES

FIGURE 1. - GEOMETRY AND BOUNDARY CONDITIONS OF CUBIC CAVITY FLOW.

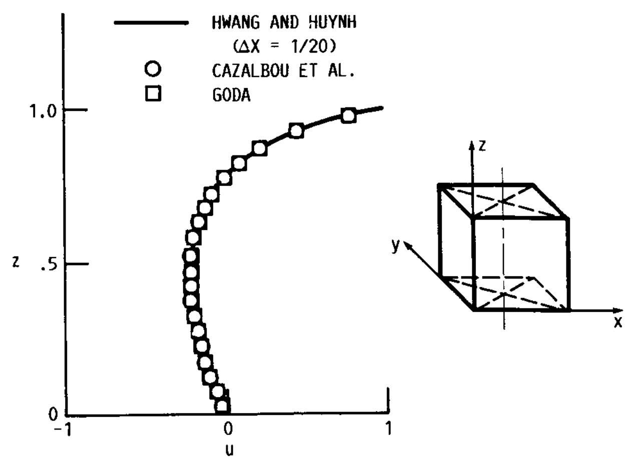


FIGURE 2. - COMPARISON OF VELOCITY PROFILES ON THE CENTERLINE FOR RE = 100.

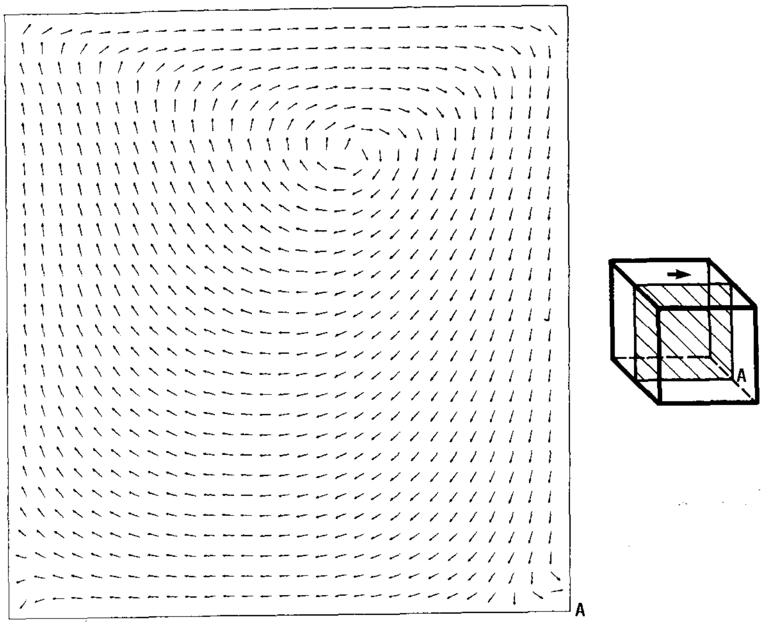


FIGURE 3. - VELOCITY FIELDS IN THE SYMMETRY PLANE FOR RE = 100, ΔX = 1/30.

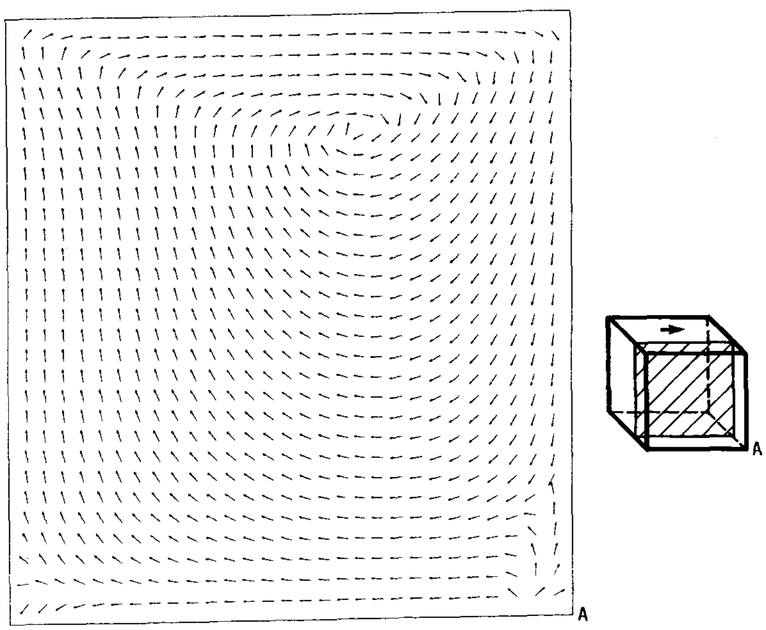
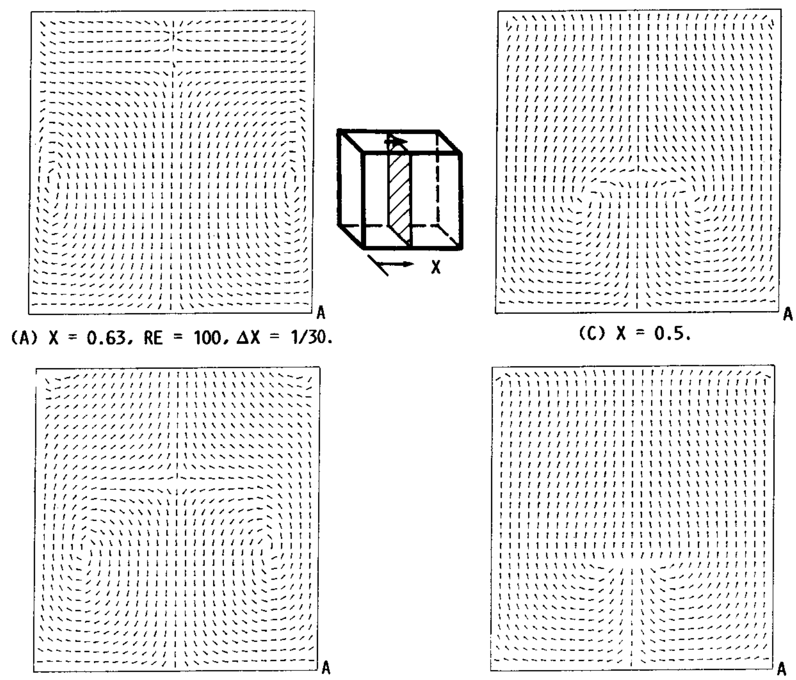


FIGURE 4. - VELOCITY FIELDS IN THE VERTICAL PLANE NEXT TO THE SIDE WALL FOR RE = 100, ΔX = 1/30.



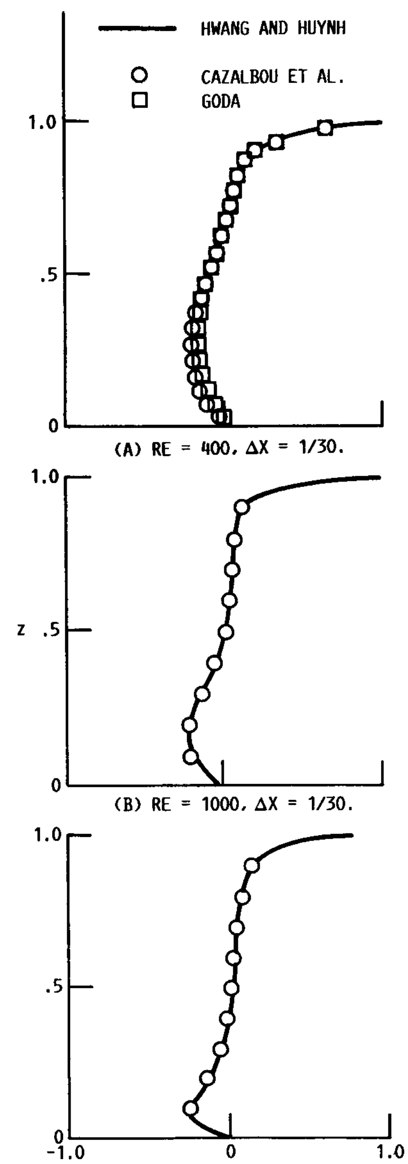
(A) X = 0.63, RE = 100, ΔX = 1/30.

(C) X = 0.5.

(B) X = 0.57.

(D) X = 0.47.

FIGURE 5. - DEVELOPMENT OF SECONDARY VORTICES.



(A) RE = 400, ΔX = 1/30.

(B) RE = 1000, ΔX = 1/30.

(C) RE = 2000, ΔX = 1/50.

FIGURE 6. - COMPARISON OF VELOCITY PROFILES ON THE CENTERLINE.

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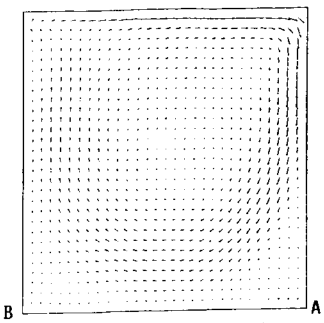
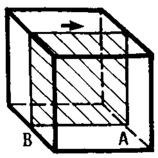
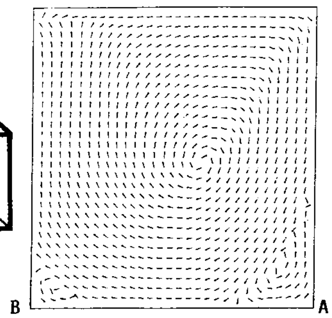
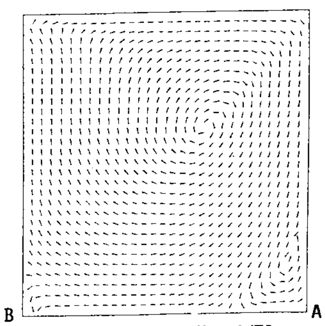


(A) RE = 400, ΔX = 1/30.

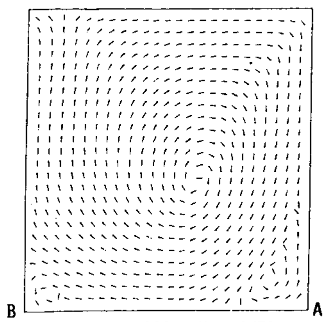




(C) RE = 1000, ΔX = 1/30.

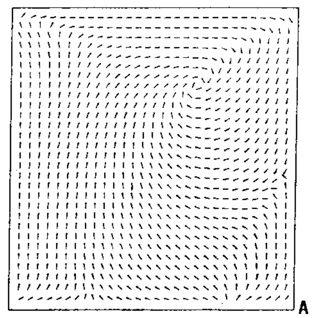


(B) RE = 400, ΔX = 1/30.

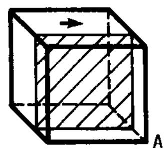


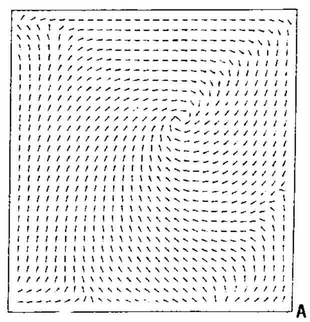
(D) RE = 2000, ΔX = 1/50.

FIGURE 7. - VELOCITY VECTORS IN THE SYMMETRY PLANE.

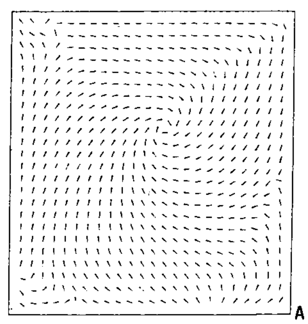


(A) RE = 400, ΔX = 1/30.



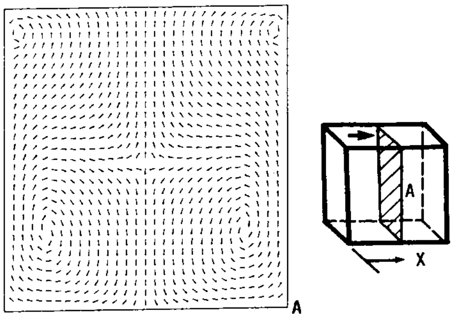


(B) RE = 1000, ΔX = 1/30.

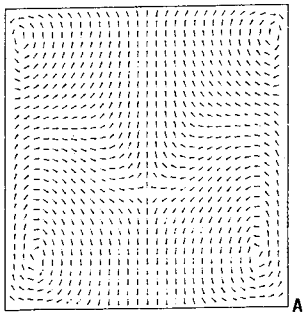


(C) RE = 2000, ΔX = 1/50.

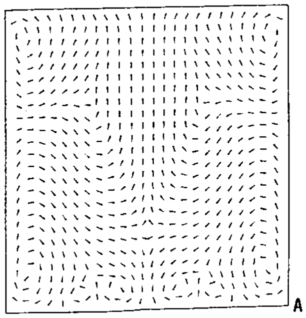
FIGURE 8. - VELOCITY FIELDS IN THE VERTICAL PLANE NEXT TO THE SIDE WALL.



(A) X = 0.57, RE = 400, ΔX = 1/30.



(B) X = 0.57, RE = 1000, ΔX = 1/30.



(C) X = 0.54, RE = 2000, ΔX = 1/50.

FIGURE 9. - SECONDARY VORTICES.

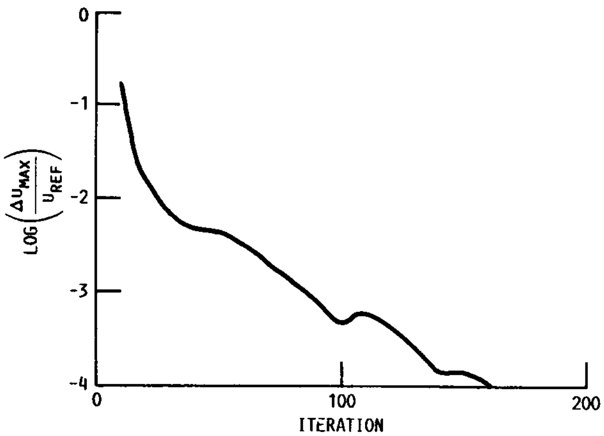


FIGURE 10. - CONVERGENCE HISTORY OF CUBIC CAVITY FLOW, RE = 100, CPU = 19.8 SEC, ΔX = 1/20.

1. Report No.	2. Government Accession No.	3. Recipient's Catalog No.
NASA TM-89851		

4. Title and Subtitle	5. Report Date
A Finite Difference Scheme for Three-Dimensional Steady Laminar Incompressible Flow	
	6. Performing Organization Code 505-40-90
7. Author(s) Danny P. Hwang and Hung T. Huynh	8. Performing Organization Report No. E-3513
	10. Work Unit No.
9. Performing Organization Name and Address National Aeronautics and Space Administration Lewis Research Center Cleveland, Ohio 44135	11. Contract or Grant No.
	13. Type of Report and Period Covered Technical Memorandum
12. Sponsoring Agency Name and Address National Aeronautics and Space Administration Washington, D.C. 20546	14. Sponsoring Agency Code

15. Supplementary Notes

Prepared for the 5th International Conference on Numerical Methods in Laminar and Turbulent Flow, cosponsored by Concordia University, P&W Canada, the International Journal for Numerical Methods in Fluids, and the Natural Sciences and Research Council of Canada, Montreal, Canada, July 6-10, 1987.

16. Abstract

A finite difference scheme for three-dimensional steady laminar incompressible flows is presented. The Navier-stokes equations are expressed conservatively in terms of velocity and pressure increments (delta form). First order upwind differences are used for first order partial derivatives of velocity increments resulting in a diagonally dominant matrix system. Central differences are applied to all other terms for second order accuracy. The SIMPLE pressure correction algorithm is used to satisfy the continuity equation. Numerical results are presented for cubic cavity flow problems for Reynolds numbers up to 2000 and are in good agreement with other numerical results.

17. Key Words (Suggested by Author(s))	18. Distribution Statement
Incompressible; Three-dimension; Navier-Stokes equations; Separated flow; Cubic cavity flow	Unclassified - unlimited STAR Category 34

19. Security Classif. (of this report)	20. Security Classif. (of this page)	21. No. of pages	22. Price*
Unclassified	Unclassified	15	A02

*For sale by the National Technical Information Service, Springfield, Virginia 22161